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Assessment Fourier coefficients of a function of class $L(p,\alpha)$

Ismet Temaj[1]

Abstract

In this paper, we'll give a necessary and sufficient condition that a function $f(x)=\sum_{n=1}^{\infty} a_n \cos nx$, where coefficient $a_n (n=1,2,...)$ are quasi-monotone, to be of class $L(p,\alpha)$.

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1 Introduction

The studying of order of decrease of Fourier coefficients of a function

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[1] Prishtina University Education Faculty, Prishtina, Kosovo,
e-mail: itemaj63@yahoo.com

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belonging to different subclasses of class L_p ($p \geq 1$) represents one of the fundamental issues of Fourier theory. This paper deals with the Fourier coefficients ($a_n \downarrow 0$ and quasi-monotone) of a function of class $L(p,\alpha)$, where $1 \leq p < \infty$, $-1 < \alpha p < p-1$.

2 Preliminary Notes

That's why, first of all, we'll represent the main statements needed for representation of the results of this paper.

Definition 2.1 *A sequence* $\{b_n\}$ *is quasi-monotone if* $b_n > 0$, *and* $n^{-\tau} b_n \downarrow 0$ *for some* $\tau > 0$.

Definition 2.2 *Let* $1 \leq p < \infty$, *we say that function* f *with period* 2π *is in class* L_p *if*

$$\|f\|_p = \left\{ \int_0^{2\pi} | f(x) |^p \, dx \right\}^{1/p} < \infty$$

So

$$L_p = \left\{ f(x) / \|f(x)\|_p = \left\{ \int_0^{2\pi} | f(x)|^p \, dx \right\}^{1/p} < \infty \right\}$$

Definition 2.3 *A function* $f(x)$ *is said to belong to the class* $L(p,\alpha)$, *if:*

$$\|f\|_{p,\alpha} = \left\{ \int_0^{\pi} |f(x)|^p (\sin x)^{\alpha p} \, dx \right\}^{1/p} < \infty$$

where $1 \leq p < \infty$, $-1 < \alpha p < p-1$.

So

$$L(p,\alpha) = \left\{ f(x) / \|f\|_{p,\alpha} = \left\{ \int_0^{\pi} |f(x)|^p (\sin x)^{\alpha p} \, dx \right\}^{1/p} < \infty \right\}$$

The following affirmation gives necessary condition receptively adequate that is necessary to complete Fourier coefficients in order that function belongs to class L_p ($L(p,\alpha)$).

Theorem 2.4 (Hausdorff-Young) [2, p. 211] *Let* $1 \le p \le 2$ *and* $q = \frac{p}{p-1}$ ($2 \le q \le \infty$). *The following estimate holds true*

1) *If* $f \in L_p$ *and* $\{c_n\}_{n=-\infty}^{\infty}$ *are Fourier coefficients of function, then*

$$\left\{ \sum_{|n|=0}^{\infty} |c_n|^q \right\}^{1/q} \le A(p) \|f\|_p$$

2) *If* $\{c_n\}_{n=-\infty}^{+\infty}$ *is sequence of numbers such that*

$$\sum_{|n|=0}^{\infty} |c_n|^p < \infty$$

then there exists function $f \in Lq$ *with Fourier coefficients* $\{c_n\}$ *the inequality*

$$\|f\|_q \le A'(q) \left\{ \sum_{n=0}^{\infty} |c_n|^p \right\}^{1/p}$$

holds true.

Theorem 2.5 *(Hardy- Littlewood)* [2, p. 657] *The necessary and sufficient condition that* $\sum_{n=1}^{\infty} a_n \cos nx$ $a_n \downarrow 0$ *be the Fourier series of a function* $f \in L_p$, $p > 1$ *is that the series* $\sum_{n=1}^{\infty} a_n^p n^{p-2} < +\infty$.

Theorem 2.6 [3] *The necessary and sufficient condition that the* $\sum_{n=1}^{\infty} a_n \cos nx$ *where* $\{a_n\}$ *is positive and quasi-monoton be Fourier series of a function*

$f \in L(p,\alpha)$, *where* $1 \le p < \infty$, $-1 < \alpha p < p-1$ *is that the series*

$$\sum_{n=1}^{\infty} (a_n)^p n^{p-\alpha p-2} < +\infty$$

In [1] given the following theorem concerning the Fourier coefficients of a function belonging to L_p class.

Theorem 2.7 [1] *Let* $f \in L_p$ ($p \ge 1$), *function given with Fourier series*

$$f(x) = \sum_{n=1}^{\infty} a_n \cos nx \quad a_n \downarrow 0$$

Then

$$\frac{S_1}{1}, \frac{S_2}{2}, \frac{S_3}{3}, \ldots$$

are also Fourier coefficients of a function of class L_p, *where* $S_n = \sum_{k=1}^{n} a_k$.

As you can see from Theorem 2.7 the connection is becoming between coefficients $\{a_n\}$ and $\{A_n\} = \left\{ \frac{1}{n} \sum_{k=1}^{n} a_n \right\}$, where $a_n \downarrow 0$ $A_n \downarrow 0$.

A question is settled down if the coefficients $\{a_n\}$ are quasi-monoton will the coefficients $\{A_n\} = \left\{ \frac{1}{n} \sum_{k=1}^{n} a_n \right\}$ be quasi-monoton. A following lemma gives the positive answer to this question.

Lemma 2.8 [4] *If* $\{a_n\}$ *is positive and quasi-monoton, then* $\{A_n\} = \left\{ \frac{1}{n} \sum_{k=1}^{n} a_k \right\}$ *is also positive and quasi-monoton.*

3 Main Results

The purpose of this paper is to reformulate the Theorem 4. in case when function $f(x) \in L(p,\alpha)$ and appropriate coefficients are quasi-monoton.

Theorem 3.1 *Let* $f(x) \in L(p,\alpha)$ *(*$1 \le p < \infty, \quad -1 < \alpha p < p-1$*), function given with Fourier series*

$$f(x) = \sum_{n=1}^{\infty} a_n \cos nx,$$

where $\{a_n\}$ *is positive and quasi-monoton. Then series*

$$\sum_{n=1}^{\infty} A_n \cos nx$$

where $\{A_n\} = \left\{ \frac{1}{n} \sum_{k=1}^{n} a_k \right\}$*, will be Fourier series of a function* $F(x)$ *of class* $L(p,\alpha)$.

Proof: Let $f(x) \in L(p,\alpha)$ $(1 \le p < \infty, \quad -1 < \alpha p < p-1$, function given with Fourier series

$$f(x) = \sum_{n=1}^{\infty} a_n \cos nx,$$

where $\{a_n\}$ is positive and quasi-monoton.

Since $\{a_n\}$ is positive and quasi-monoton and due to Lemma 2.8

$$\{A_n\} = \left\{ \frac{1}{n} \sum_{k=1}^{n} a_k \right\}$$

is positive and quasi-monoton. To proof theorem we have to show that $\sum_{n=1}^{\infty} (A_n)^p n^{p-\alpha p-2} < +\infty$, then by Theorem 2.6 follows that series $\sum_{n=1}^{\infty} A_n \cos nx$ is Fourier series of a function $F(x)$ of class $L(p,\alpha)$.

Let

$$f_1(x)=\int_0^x f(x)dx \quad \text{and} \quad f_2(x)=\int_0^x f_1(x)dx$$

Then

$$f_2(x)=\sum_{k=1}^{\infty} a_k[1-\cos kx]\cdot k^{-2} \geq \sum_{k=1}^{n} a_k[1-\cos kx]k^{-2}$$

for

$$\frac{\pi}{4(n+1)} \leq x \leq \frac{\pi}{4n}$$

we have

$$f_2(x) \geq B_1 \cdot n^{-2} \cdot \sum_{k=1}^{n} a_k \geq B_1 \cdot n^{-1} A_n$$

for same constant B_1. So $A_n \leq B \cdot n \cdot f_2(x)$ for same constant B.

Thus:

$$\sum_{n=1}^{\infty} n^{p-\alpha p-2}[A_n]^p \leq B \cdot \sum_{n=1}^{\infty} n^{p-\alpha p-2} \cdot n^p [f_2(x)]^{p} =$$

$$= B \cdot \sum_{n=1}^{\infty} n^{2p-\alpha p-2} \min_{\frac{\pi}{4(n+1)} \leq x \leq \frac{\pi}{4n}} [f_2(x)]^p \leq$$

$$\leq B \cdot \sum_{n=1}^{\infty} \int_{\frac{\pi}{4(n+1)}}^{\frac{\pi}{4n}} (\sin x)^{\alpha p-p} \cdot \left[\frac{f_2(x)}{x}\right]^p dx =$$

$$= B \cdot \int_0^{\pi/4} (\sin x)^{\alpha p-p} \cdot \left[x^{-1} f_2(x)\right]^p dx \leq B(\alpha,p) \cdot \int_0^{\pi/4} (\sin x)^{\alpha p-p} \cdot \left[x^{-1} f_1(x)\right]^p dx$$

$$\leq B(\alpha,p) \cdot \int_0^{\pi/4} (\sin x)^{\alpha p} \cdot \left(f(x)\right)^p dx < \infty$$

A similar method may be used to estimate

$$\int_{\pi/4}^{\pi} (\sin x)^{\alpha p} \cdot \left(f(x)\right)^{p} dx < \infty$$

So

$$\sum_{n=1}^{\infty} n^{p-\alpha p-2} [An]^{p} < \infty .$$

This finishes the proof of Theorem 3.1. □

The question appears: Is the converse valuable of Theorem 2.7 and 3.1 if the series $\sum_{n=1}^{\infty} A_n \cos nx$ is Fourier series, will Fourier series be $\sum_{n=1}^{\infty} a_n \cos nx$. From the following example it is proved that the converse of Theorem 2.7 and 3.1 doesn't worth.

Example 3.2 Let

$$\sum_{n=1}^{\infty} A_n \cos nx = \sum_{n=1}^{\infty} \frac{1}{\mathrm{n}} (-1)^{n} \cos nx$$

We have

$$\left[\sum_{n=1}^{\infty} |A_n|^{p}\right]^{\frac{1}{p}} = \left[\sum_{n=1}^{\infty} \left|\frac{1}{n}(-1)^{n}\right|^{p}\right]^{\frac{1}{p}} = \left\{\sum_{n=1}^{\infty} \frac{1}{n^{p}}\right\}^{\frac{1}{p}} < \infty, \qquad 1 < p \le 2.$$

Hence by the theorem1.(Hausdorf-Young), A_n is the Fourier coefficients of a function $F(x) \in L_q$, where $\frac{1}{p} + \frac{1}{q} = 1$, $1 \le p < 2$, $q \ge 2$.

Now, if $\sum_{n=1}^{\infty} a_n \cos nx$ Fourier series of a function e $f(x) \in L_p$, then we have by Theorem 2.4 (Hausdorf – Young) necessarily $\left(\sum_{n=1}^{\infty} | a_n |^{q}\right)^{\frac{1}{q}} < \infty$, where $\frac{1}{p} + \frac{1}{q} = 1$, $1 < p \le 2$, $q \ge 2$.

But $A_n = \frac{1}{n}\sum_{k=1}^{n} a_k = \frac{1}{n}(-1)^n$ so follow

$$S_n = \sum_{k=1}^{n} a_k = (-1)^n ,$$

$$a_n = S_n - S_{n-1} = (-1)^n - (-1)^{n-1} = (-1)^{n-1}(-1-1) = 2\cdot(-1)^n$$

$$\left(\sum_{n=1}^{\infty} |a_n|^q\right)^{\frac{1}{q}} = \left(\sum_{n=1}^{\infty} |2(-1)^n|^q\right)^{\frac{1}{q}} = \sum_{n=1}^{\infty}(2^q)^{\frac{1}{q}} = \sum_{n=1}^{\infty} 2 = \infty .$$

Therefore $\sum_{n=1}^{\infty} a_n \cos nx = \sum_{n=1}^{\infty} 2\cdot(-1)^n \cos nx$ is not the Fourier series of a function $f(x) \in L_p$.

So the question is settled down. What conditions of coefficients a_n will be fulfilled in order that converse is valuable. A following theorem gives answer to the question.

Theorem 3.3 *Let*

$$f(x) \approx \sum_{n=1}^{\infty} a_n \cos nx ,$$

where $\{a_n\}$ *is positive and quasi-monoton. Then a necessary and sufficient condition that* $\sum_{n=1}^{\infty} a_n \cos nx$ *be the Fourier series of function* $f(x) \in L(p,\alpha)$ *is that:*

$$\sum_{n=1}^{\infty} A_n \cos nx$$

to be the Fourier series of a function $F(x)$ *be belonging to* $L(p,\alpha)$ *where* $1 \le p < \infty, \quad -1 < \alpha p < p-1$ *and* $A_n = \frac{1}{n}\sum_{k=1}^{n} a_k$.

Proof: The necessary part follows from Theorem 3.1 as a particular case.

Sufficiency. Suppose that series $\sum_{n=1}^{\infty} A_n \cos nx$ is Fourier series of a function $F(x) \in L(p,\alpha)$. Since $\{a_n\}$ is positive and quasi-monoton, then by Lemma 2.8 follows $\{A_n\}$ is positive and quasi-monoton. Hence by Theorem 2.6 we have

$$\sum_{n=1}^{\infty} n^{p-\alpha p-2} A_n^p < \infty .$$

Since sequence $\{a_n\}$ is positive and quasi-monoton then for some constant $\tau > 0$, sequence $n^{-\tau} a_n \downarrow 0$, and fore some constant $B_1 > 0$ we have $n^{-\tau} a_n \le B_1 k^{-\tau} a_k$ for $k < n$, then it follows that

$$A_n = \frac{1}{n}\sum_{k=1}^{n} a_k = \frac{1}{n}\sum_{k=1}^{n} k^{-\tau} a_k k^{\tau} \ge \frac{1}{B_1}\frac{1}{n}\cdot n^{-\tau} a_n \sum_{k=1}^{n} k^{\tau} =$$

$$= \frac{1}{B_1}\frac{1}{n} n^{-\tau} a_n \cdot n n^{\tau} = \frac{1}{B_1} a_n$$

$$a_n \le B_1 \cdot A_n$$

So that

$$\sum_{n=1}^{\infty} n^{p-\alpha p-2} a_n^p \le (B_1)^p \sum_{k=1}^{\infty} n^{p-\alpha p-2} A_n^p < \infty .$$

Hence by Theorem 2.6 $f(x) \in L(p,\alpha)$ and consequently $\sum_{n=1}^{\infty} a_n \cos nx$ is the Fourier series of function $f(x)$. □

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